Minimax Linear Optimal Control of Positive Systems

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Abstract

We present a novel class of minimax optimal control problems with positive dynamics, linear objective function and linear constraints. This class can be analyzed with dynamic programming. To solve this class of problems we use standard value iteration and give an explicit solution to the Bellman equation of a given problem inside our novel class. Moreover, the sparsity of the matrix determining the control policy is inherited from the constraint matrix of the problem statement. This simplifies the implementation in large systems.

Background: Positive systems and minimax optimal control

Minimax optimal control problems aim to design control systems that minimize the worst-case outcomes, in uncertain environments.. These problems are often computationally challenging. Dynamic programming offers approaches such as value Iteration to solve this problems breaking them into simpler sub-problems.

Positive systems are dynamical models where the state and output values remain nonnegative as long as the initial states and inputs are also nonnegative. These systems are particularly useful when modelling various real-world phenomena, and their stability can be easily verified using linear Lyapunov functions.

Together, these elements offer a robust and tractable framework for designing control systems.

Main Result

Theorem 1. *Let* $A \in \mathbb{R}^{n\times n}$, $B = [B_1 ... B_m] \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times l}$, $E = \left[E_1^\top ... E_m^\top\right]^\top \in \mathbb{R}_+^{m\times n}$, $G = \left[G_1^\top, ..., G_l^\top\right]^\top \in \mathbb{R}_+^{l\times n}$, $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\gamma \in \mathbb{R}^l$.

Suppose that

$$\max_{|w|\leq Gx} \left[s^\top x + r^\top u - \gamma^\top w\right] \geq 0$$

when $x \geq 0$ *and* $|u| \leq Ex$. *Suppose also that*

$$A \geqslant |B|\, E + |F|\, G \tag{2}$$

$$s \geqslant E^\top |r| - G^\top |\gamma| \tag{3}$$

Then the following statements are equivalent.

(i) The optimal control problem (1), has a finite value for every $x_0 \in \mathbb{R}_+^n$

(ii) The recursive sequence $\{p_k\}_{k=0}^{\infty}$ *with* $p_0 = 0$ *and*

$$p_k = s + A^\top p_{k-1} - E^\top \left|r + B^\top p_{k-1}\right| + G^\top \left|-\gamma + F^\top p_{k-1}\right| \tag{4}$$

has a finite limit.

(iii) There exists $p \in \mathbb{R}_+^n$ *such that*

$$p = s + A^\top p - E^\top \left|r + B^\top p\right| + G^\top \left|-\gamma + F^\top p\right|. \tag{5}$$

If (iii) is true then (1) has the minimal, finite, non-negative value $p^T x_0$, *with p being the limit of the recursive sequence* $\{p_k\}_{k=0}^{\infty}$ *in* (ii). *Moreover, the control law* $u(t) = -Kx(t)$ *is optimal when*

$$K := \begin{bmatrix} \text{sign}(r_1 + p^\top B_1)E_1 \\ \vdots \\ \text{sign}(r_m + p^\top B_m)E_m \end{bmatrix}. \tag{6}$$

Remark: *In (6), it can be observed that the structure of the control gain K is directly determined by the E matrix. Therefore, it inherits its sparsity structure.*

Optimal Problem

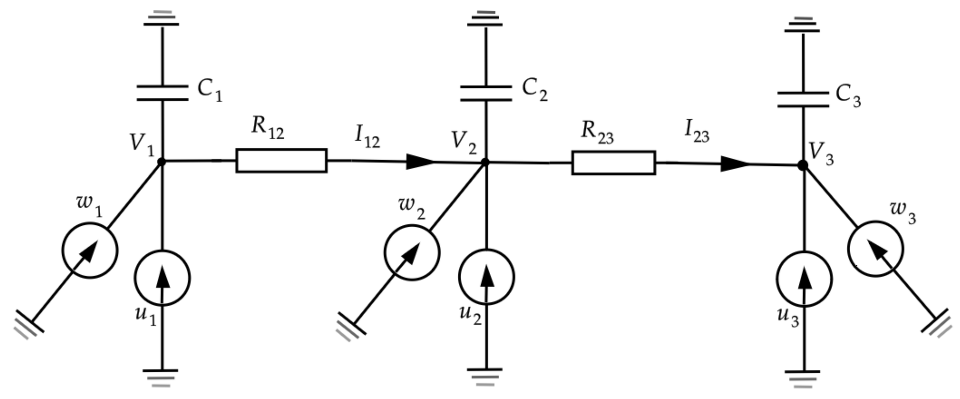
Discrete time, infinite horizon, minimax optimal control problem with non-negative cost and positive dynamics with linear objective function and constraints.

$$\inf_{\mu} \max_{w} \sum_{t=0}^{\infty} \left[s^\top x(t) + r^\top u(t) - \gamma^\top w(t)\right] \tag{1}$$

subject to

$$x(t+1) = Ax(t) + Bu(t) + Fw(t),$$
$$u(t) = \mu(x(t)) \;;\; x(0) = x_0$$
$$|u| \leq Ex \;;\; |w| \leq Gx$$

Example: Optimal Voltage Control in a DC Power Network



Discretized Voltage dynamics

$$C\left(\frac{V(\tau h + h) - V(\tau h)}{h}\right) = -\mathcal{L}_{\mathcal{R}} V(\tau h) + u(\tau h) + w(\tau h).$$

$$[\mathcal{L}_{\mathcal{R}}]_{i,j} = \begin{cases} -\frac{1}{R_{i,j}} & \text{if } i \neq j \\ \sum_{j=1}^{n} \frac{1}{R_{i,j}} & \text{if } i = j \end{cases}$$

$$x(t+1) = \left[I - hC^{-1} \cdot \mathcal{L}_{\mathcal{R}}\right] x(t) + hC^{-1}u(t) + hC^{-1}w(t)$$

- The network structure can be encoded in the constraint matrix E.
- A closed-form minimax optimal controller is easily derived

Conclusion

The presented problem can be applied to large and scalable dynamical systems efficiently, and the sparsity of the control policy matrix is directly related to the matrix of the problem statement.

Future Work

- There is still room for improving our methodology. Value iteration is computationally expensive, more efficient methods are needed.
- Analysis of the limitations of the disturbance bounds.

References

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