Linear Regulator-based synchronization of positive discrete-time multi-agent systems

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1. MOTIVATION

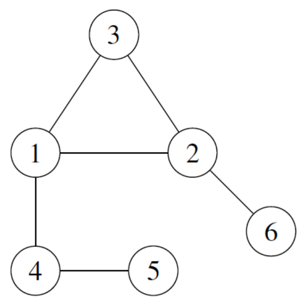
A fundamental challenge in control theory is designing protocols that achieve synchronization in interconnected systems. This work introduces a static feedback protocol derived from the Linear Regulator problem, where the stabilizing policy is determined by solving an algebraic equation using a linear program. Necessary and sufficient conditions are established to guarantee the positivity of each agent's trajectory for all nonnegative initial conditions.

2. MODEL DESCRIPTION

Graph Description

- Weighted, undirected, connected graphs

  $\mathcal{G} \in \mathbb{G},\ \mathcal{G} = \{\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}\},\ N = |\mathcal{V}_\mathcal{G}|,\ \mathcal{E}_\mathcal{G} \subset \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$



- The regularised Laplacian $\mathcal{L}$ of $\mathcal{G}$ is

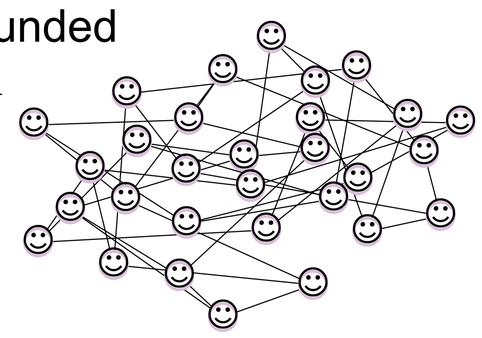
$$\mathcal{D} = I - (I + D)^{-1}\mathcal{L}$$

where D is the degree matrix of the graph and $\mathcal{D}\mathbf{1} = \mathbf{1}$.

- We consider graph families for which the regularized Laplacian eigenvalues $\mu_i(\mathcal{G})$ are lower and upper bounded

$$\mathbb{G}_{[\gamma,\beta]} = \{\mathcal{G} \in \mathbb{G} \,|\, \gamma \leq \mu_i(\mathcal{G}) \leq \beta,\ \forall i > 1\}$$

where $\gamma \in (-1, \beta],\ \ \beta \in (0, 1)$.



Multi-agent systems

- Arbitrary number of homogeneous, LTI **positive** agents

$$x_i(t+1) = Ax_i(t) + Bu_i(t); \quad A \in \mathbb{R}_+^{n\times n}, B \in \mathbb{R}^{n\times m}. \qquad (1)$$

- The communication network is described by a graph
- Agents access the relative information of their neighbours through full-state measurements

$$\zeta_i(k) = \frac{1}{1+\sum_{j=1}^N w_{ij}} \sum_{k\in\mathcal{N}_i} w_{ik}(x_i(t) - x_k(t)) \qquad (2)$$

where u_i is a function of ζ_i .

3. PRELIMINAIRES

State synchronization

- The agents in the network achieve state synchronization if

$$\lim_{t\to\infty} [x_i(t) - x_j(t)] = 0, \qquad \forall i, j \in \{1, \ldots, N\}.$$

- The *state synchronisation problem* consists of finding, if possible, a linear static protocol

$$u_i(t) = F\zeta_i(t) \quad i = 1, \ldots, N$$

such that for any graph in the family and all initial conditions state synchronization is achieved.

This problem is refer to as the *positive consensus problem* if, for any selection of nonnegative initial conditions, the states of the agents remain nonnegative.

- The multi-agent system $(1), (2)$ achieves state synchronization if

$$\tilde{\eta}_i(t+1) = (A + (1-\mu_i)BF)\tilde{\eta}_i(t), \quad i = 2, \ldots, N$$

are globally asymptotically stable, where

$$\eta := (T^{-1} \otimes I_n)x = (\eta_1, \ldots, \eta_N)^\top,\ \ \mathcal{D} = T\Lambda_\mathcal{D}T^{-1},\ \ \Lambda_D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

and the synchronized trajectory is given by

$$x_s(t) = A^t \frac{1}{N} \sum_{i=1}^N x_i(0).$$

Linear Regulator

Let $\tilde{A} \in \mathbb{R}_+^{n\times n},\ \tilde{B} \in \mathbb{R}^{n\times m},\ \tilde{E} \in \mathbb{R}_+^{m\times n}$ and $s \in \mathbb{R}_+^n$. Suppose $\tilde{A} - |\tilde{B}|E \geq 0$. Then the following problem has a finite value for every nonnegative initial condition

$$\inf_\mu \sum_{t=0}^\infty [s^\top x(t)]$$

$$\text{Subject to}$$

$$x(t+1) = \tilde{A}x(t) + \tilde{B}u(t),\ x(0) = x_0$$

$$u(t) = \mu(x(t)),\ |u| \leq \tilde{E}x$$

if and only if there exists a vector $p \in \mathbb{R}_+^n$ such that $p = s + \tilde{A}^\top p - \tilde{E}^\top |\tilde{B}^\top p|$.

The optimal control law is $K := \operatorname{diag}\big(\operatorname{sign}(\tilde{B}^\top p)\big)\tilde{E}$. (3)

- If $s > 0$ and there exists $u = -Kx$ with $|u| \leq Ex$ such that $\tilde{A} - \tilde{B}K$ is Schur stable, then the vector p maximises the linear program

$$\text{Maximize } \mathbf{1}^\top p \text{ over } p \in \mathbb{R}_+^n,\ \zeta \in \mathbb{R}_+^m$$

$$\text{Subject to } p \leq s + \tilde{A}^\top p - \tilde{E}^\top \zeta$$

$$-\zeta \leq \tilde{B}^\top p \leq \zeta.$$

4. MAIN RESULTS

Linear Regulator-based protocol

Consider the multi-agent system $(1), (2)$ with $A \in \mathbb{R}_+^{n\times n},\ B \in \mathbb{R}^{n\times m}$. Let $E \in \mathbb{R}_+^{m\times n}, s \in \mathbb{R}_+^n$ such that $s > 0$ and $\mathcal{D}$ be the row stochastic matrix associated with a graph $\mathcal{G} \in \mathbb{G}_{[\gamma,\beta]}$ with N agents. Suppose

$$A - (1-\gamma)|B|E \geq 0$$

$\gamma \in (-1, 1)$. The LR-based protocol is given by

$$u_i = -\rho K\zeta_i, \qquad (4)$$

where $\rho \geq \dfrac{1}{1-\beta},\ \ \beta \in (0,1)$ and K satisfies (3) with

$$\tilde{A} = A,\ \tilde{B} = B,\ \tilde{E} = \frac{1}{\rho}E.$$

Positive synchronization

Consider a graph family $\mathcal{F} \subseteq \mathbb{G}_{[\gamma,\beta]}$ and the multi-agent system (1), (2). Suppose there exists $u = -Kx$ with $|u| \leq Ex$ and $\tilde{A} - \tilde{B}K$ is Schur. Recall the protocol (4). Suppose also that

$$-1 < \gamma \leq \frac{1}{1+\sum_{j=1}^N w_{ij}} \quad \forall i = 1, \ldots N.$$

The trajectories of the multi-agent system remain nonnegative for all nonnegative initial conditions if and only if BK is nonnegative.

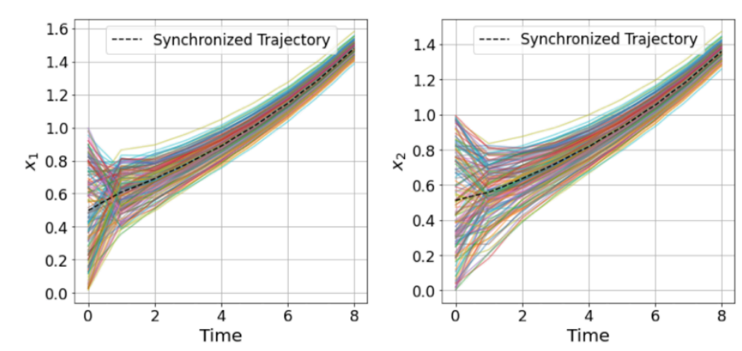
5. SIMULATIONS

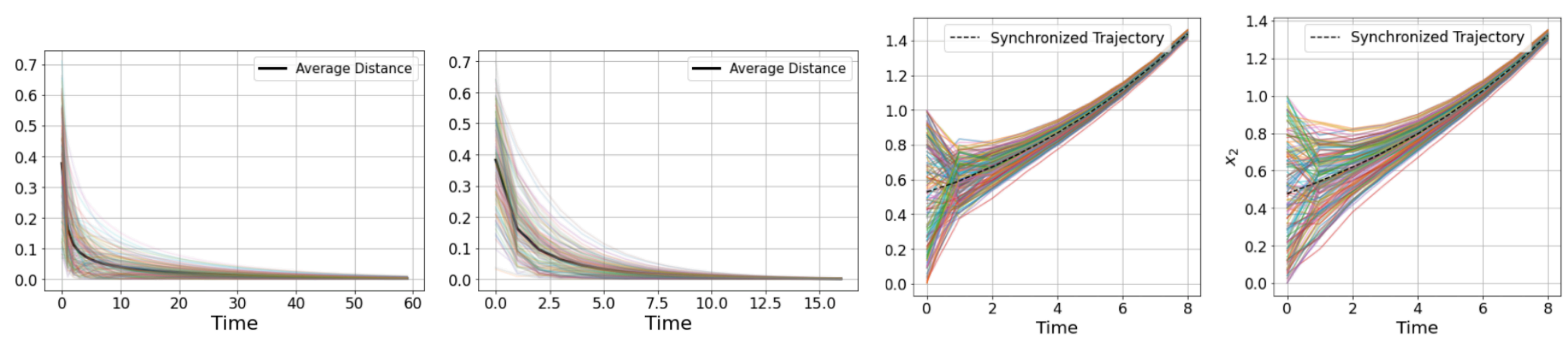
Consider multi-agent systems composed by 150 homogeneous, positive agents and the family of regular graphs $\mathcal{G} \subseteq \mathcal{F}_R \subseteq \mathbb{G}_{[\gamma,\beta]}$ with degree 7 and 20.

$$A = \begin{bmatrix} 0.4 & 0.8 \\ 0.4 & 0.7 \end{bmatrix};\ B = \begin{bmatrix} -0.6 & 0.002 & -0.2 \\ -0.4 & 0.005 & 0.03 \end{bmatrix};$$

$$\beta = 0.25,\ \rho = 3.83,\ \tilde{E} = \frac{1}{\rho}E = \begin{bmatrix} 0.02 & 0.05 \\ 0.07 & 0.07 \\ 0.07 & 0.003 \end{bmatrix}$$

$s = \mathbf{1}$.





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